# Global existence of Navier-Stokes-Vlasov-Fokker-Planck equations

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- Introduction to models

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# Introduction to models

1. Interaction of fluids and particles : aerosol and spray
2. fluid-particle interaction : the gravity settling process or centrifugal forces : applications in biotechnology medicine, waste-water recycling, Diesel engines and rocket propulsors etc.
3. The equations : Combination of fluid equations and Fokker-Planck(or Vlasov) equations
4. There are two important regimes :
   - Flowing regime : $\rho_p \sim \rho_f$
   - Bubbling regime : $\rho_p \ll \rho_f$

# Derivation of the equations(Caflisch-Papanicolaou, '83)

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$$\begin{cases} \phi_t + \nabla \cdot (\phi u_p) = 0, \\ (1-\phi)_t + \nabla \cdot ((1-\phi) u_f) = 0, \\ \rho_p \phi (u_{pt} + (u_p \cdot \nabla) u_p) = -\phi \nabla p - \phi S(u_p - u_f), \\ \rho_f (1-\phi)(u_{ft} + (u_f \cdot \nabla) u_f) = (1-\phi)(\Delta u_f - \nabla p_f) + \phi S(u_p - u_f). \end{cases}$$

- $u_p$, $u_f$ : average velocities of the particles and fluids, $\phi$ : volume fraction density of the particles

# Derivation of the equations(Caflisch-Papanicolaou, '83)

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  - $u_p$, $u_f$ : average velocities of the particles and fluids, $\phi$ : volume fraction density of the particles
  - From the two fluid model, $u_p$ equations are replaced by kinetic equations
  - Ignoring gravity effect and assuming the volume fraction density is constant (compressible or incompressible) Navier-Stokes(or Euler)+Vlaosv-(Fokker-Planck) equations

# Classification of sprays('81, P.J. O'Rourke)

- Very thin spray model( compressible Euler and Vlasov )

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u_f) = 0, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p = 0, \\ \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (Ff) = 0. \end{cases}$$

$p = p(\rho)$ and $F = -S(v-u)$

# Classification of sprays('81, P.J. O'Rourke)

- Thin spray model( compressible Euler and Vlasov )

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u_f) = 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p = -\int Ffdv, \\ \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (Ff) = 0. \end{cases}$$

$p = p(\rho)$ and $F = -S(v - u)$

# Classification of sprays('81, P.J. O'Rourke)

- Moderately thick spray model( compressible Euler and Vlasov )

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u_f) = 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p = -\int F f dv, \\ \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F f) = Q(f). \end{cases}$$

$p = p(\rho)$ and $F = -S(v-u)$. $Q(f)$ : collision kernel.

# Classification of sprays('81, P.J. O'Rourke)

- Thick spray model( compressible Euler and Vlasov )

$$\begin{cases} \partial_t(\alpha\rho) + \nabla\cdot(\alpha\rho u_f) = 0, \\ \partial_t(\alpha\rho u) + \nabla\cdot(\alpha\rho u\otimes u) + \nabla p = -\int Ffdv, \\ \partial_t f + v\cdot\nabla_x f + \nabla_v\cdot(Ff) = Q(f). \end{cases}$$

$p = p(\rho)$ and $F = -S(v-u)$. $Q(f)$ : collision kernel and $1-\alpha = \int fdv$

# Navier-Stokes + Vlasov-Fokker-Planck

• Microscopic description of the particles : particles are described by the probability density $f(t, x, v)$ by transport equation with friction force $F$

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F f - \sigma \nabla_v f) = 0,$$

• Macroscopic fluid equations; The particles are dispersed in a fluid described by its velocity field $u(t, x)$ satisfying the incompressible Navier-Stokes equations,

$$\partial_t u + (u \cdot \nabla) u + \nabla p - \nu \Delta u = -\int_{\mathbb{R}^d} F f dv, \qquad \nabla \cdot u = 0.$$

# Thin spray model

$F = F_0(u - v)$ : friction force is proportional to the relative velocity

$$-\int_{\mathbb{R}^d} Ffdv = F_0 \int_{\mathbb{R}^d} f(v-u)dv.$$

- Navier-Stokes-Vlasov-Fokker-Planck equations

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla p - \int_{\mathbb{R}^d}(v-u)fdv = 0, \quad \mathrm{div}\, u = 0, \\ \partial_t f + (v\cdot\nabla_x)f + \nabla_v\cdot((u-v)f - \nabla_v f) = 0, \end{cases} \tag{1}$$

# Brief History on the analytic study on NSVFP and related equations

1) Hamdache('98) : construction of the weak solution to the Vlasov-Stokes system in a bounded domain with specular reflection boundary conditions.
2) Boudin-Desvilletes-Grandmont-Moussa('08): construction of the global weak solution for 3D incompressible Navier-Stokes-Vlasov equations in a torus
3) Mellet-Vasseur('07) : the existence of global weak solution to compressible NSVFP equations in a bounded domain with Dirichlet or reflection boundary conditions.
4) Baranger-Desvillettes('06) : the local existence of the compressible Vlasov-Euler equations

5) Goudon-He-Moussa-Zhang('08) : Stability of solution near Maxwellian, which is equilibrium solution of the form $(u = 0, f = Me^{-\frac{|v|^2}{2}})$ in a three dimensional torus.

6) He('10) : a perturbation of the steady state of the system is globally stable for arbitrary initial data converging toward steady state with the exponential rate under specific assumptions.

7) Goudon-Jabin-Vasseur('04) : hydrodynamic limit of the global weak solution of the system (1)

8) Carrillo-Duan-Moussa('10) : Global existence of classical solutions close to equilibrium to the 3D Vlasov-Euler-Fokker-Planck equations

# The global existence of “strong” solution in 2D

- Assumptions
$\langle v\rangle^k f_0,\ \langle v\rangle^k \nabla_x f_0 \in L^p(\mathbb{R}^2\times\mathbb{R}^2)$ , $\quad \nabla u_0 \in L^p(\mathbb{R}^2)$, with $p\in(2,\infty)$, $k>3-\frac{2}{p}$ and $\langle v\rangle = (1+|v|^2)^{1/2}$

**Theorem(M. Chae, K. Kang and L., ’11)**

$\exists\ (f,\ u)$ such that

$$\begin{aligned} &\nabla u \in L^\infty(0,T;L^p(\mathbb{R}^2)),\\ &|\nabla u|^{\frac{p}{2}} \in L^2(0,T;H^1(\mathbb{R}^2)),\\ &\langle v\rangle^k \nabla_x f \in L^\infty(0,T;L^p(\mathbb{R}^2\times\mathbb{R}^2)),\\ &\langle v\rangle^{pk/2}|\nabla_x f|^{\frac{p-2}{2}}\nabla_v\nabla_x f \in L^2(0,T;L^2(\mathbb{R}^2\times\mathbb{R}^2)). \end{aligned} \tag{2}$$

# ingredient of proof

High moment estimates

D. Chae's proof of the proof of classical solution to the partial viscous Boussinesq system : Use of Brezis-Wainger's inequality

# Existence results for the incompressible fluid-kinetic equations

- Proposition

$$\|\langle v\rangle^k f_0\|_{L^p(\mathbb{R}^2\times\mathbb{R}^2)} < \infty, \qquad k > \frac{2\epsilon}{p(1-\epsilon)}, \quad p \geq 2,$$

where $\langle v\rangle = (1+|v|^2)^{\frac{1}{2}}$ and $\epsilon \in (0,\,1)$, then $f$ satisfies

$$\langle v\rangle^k f \in L^\infty(0,\ T; L^p(\mathbb{R}^2\times\mathbb{R}^2))$$

$$\langle v\rangle^{\frac{pk}{2}} \nabla_v |f|^{\frac{p}{2}} \in L^2(0,\ T; L^2(\mathbb{R}^2\times\mathbb{R}^2)).$$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\left(\frac{|x|^2}{2}+\frac{|v|^2}{2}+|\log f|\right)dvdx+\int_{\mathbb{R}^d}\frac{|u|^2}{2}dx$$
$$+\int_0^t\int_{\mathbb{R}^d}|\nabla_x u|^2 dxds+\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|(u-v)f-\nabla_v f|^2}{f}dvdxds$$
$$\leq C\left(t,\mathcal{E}(f_0,u_0),\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|x|^2 f_0 dvdx\right). \tag{3}$$

$$\frac{d}{dt}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^p dvdx+\frac{4(p-1)}{p}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\nabla_v f^{\frac{p}{2}}|^2 dvdx$$
$$=d(p-1)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^p dvdx.$$
$$\|f(t)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\leq C(t,\|f_0\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}).$$

# A priori estimates

Multiplying $\langle v\rangle^{kp} f^{p-1}$

$$\frac{1}{p}\frac{d}{dt}\|\langle v\rangle^{k} f\|^{p}_{L^p(\mathbb{R}^2\times\mathbb{R}^2)} + C_p\|\langle v\rangle^{\frac{pk}{2}}\nabla_v|f|^{\frac{p}{2}}\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}^2)}$$

$$= -\frac{1}{p}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(u\cdot\nabla_v)f^p\langle v\rangle^{kp}\,dvdx + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\nabla_v\cdot(vf)\langle v\rangle^{kp}f^{p-1}\,dxdv$$

$$+\frac{1}{p}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f^p\nabla_v^2\langle v\rangle^{kp}\,dvdx := J_{11} + J_{12} + J_{13}.$$

# A priori estimates

$$J_{11} \leq C \int_{\mathbb{R}^2} |u| \left( \int_{\mathbb{R}^2} \langle v \rangle^{kp} f^{p \cdot \frac{kp}{kp-\epsilon}} dv \right)^{\frac{kp-\epsilon}{kp}} \left( \int_{\mathbb{R}^2} \frac{1}{\langle v \rangle^{kp \cdot \frac{(1-\epsilon)}{\epsilon}}} dv \right)^{\frac{\epsilon}{kp}} dx$$

$$\leq C \|u\|_{L^{\frac{kp}{\epsilon}}(\mathbb{R}^2)} \|\langle v \rangle^k f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{kp-\epsilon}{k}} \|f\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{\epsilon}{k}}$$

$$\leq C \|u\|_{L^2(\mathbb{R}^2)}^{\frac{2\epsilon}{kp}} \|\nabla u\|_{L^2(\mathbb{R}^2)}^{1-\frac{2\epsilon}{kp}} \|\langle v \rangle^k f\|_{L^p(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{kp-\epsilon}{k}} \|f\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{\epsilon}{k}}.$$

# Existence results for the incompressible fluid-kinetic equations

$$\partial_t \omega - \Delta \omega + (u \cdot \nabla)\omega$$

$$= -\int_{\mathbb{R}^2} (v \times \nabla_x) f \, dv - \nabla_x \times (nu),$$

where $\omega = \partial_1 u_2 - \partial_2 u_1$ and $n(x,t) = \int_{\mathbb{R}^2} f(t, x, v) dv$.

# Existence results for the incompressible fluid-kinetic equations

$$\frac{1}{p}\frac{d}{dt}\|\omega\|^p_{L^p(\mathbb{R}^2\times\mathbb{R}^2)} + C_p\|\nabla|\omega|^{\frac{p}{2}}\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}^2)} + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f|\omega|^p dvdx$$

$$\leq C\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|v||\nabla_x f||\omega|^{p-1}dvdx + C\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|u||\nabla_x f||\omega|^{p-1}dvdx$$

$$:= J_{22} + J_{23}.$$

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$$:= J_{22} + J_{23}.$$

To close the estimate, we need Brezis-Weinger inequality

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq C(1+\|\nabla u\|_{L^2(\mathbb{R}^2)})(1+\log^+\|\nabla u\|_{L^p(\mathbb{R}^2)})^{\frac{1}{2}} + C\|u\|_{L^2(\mathbb{R}^2)}, \tag{4}$$

# Existence of weak solution

**Theorem**

*Let $d = 2$ or $3$. Suppose $(f_0, u_0)$ satisfies*

$$f_0 \geq 0, \; f_0 \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d),$$
$$\int_{\mathbb{R}^d} (|x|^2 + |v|^2 + |\log f_0|) f_0 dv \in L^1(\mathbb{R}^d), \; u_0 \in \mathcal{H}(\mathbb{R}^d).$$

*Then $\exists$ $(f, u)$ of incompressible Navier-Stokes-Fokker-Planck equations with $(f_0, u_0)$ such that*

$$u \in L^\infty(0, T; \mathcal{H}) \cap L^2(0, T; \mathcal{V}_\sigma(\mathbb{R}^d)) \cap \mathcal{C}^0(0, T; \mathcal{V}'(\mathbb{R}^d))$$
$$f \in L^\infty(0, T; L^\infty \cap L^1(\mathbb{R}^d \times \mathbb{R}^d)) \cap C(0, T; L^1(\mathbb{R}^d \times \mathbb{R}^d))$$
$$f|v|^2 \in L^\infty(0, T; L^1(\mathbb{R}^d \times \mathbb{R}^d)).$$

# Notation

- Notation

$$\partial^\alpha_\beta f = \partial^\alpha_x \partial^\beta_v f, \qquad \|f\|_{W^{m,p}_x L^p_v(\mathbb{R}^2\times\mathbb{R}^2)} = \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{L^p(\mathbb{R}^2\times\mathbb{R}^2)},$$

$$\|f\|_{W^{N,p}_k(\mathbb{R}^2\times\mathbb{R}^2)} = \sum_{|\alpha|+|\beta|\le N} \|\langle v\rangle^k \partial^\alpha_\beta f\|_{L^p(\mathbb{R}^2\times\mathbb{R}^2)}.$$

# Higher regularity

**Theorem**

$$u_0 \in W^{N,p}(\mathbb{R}^2), \qquad f_0 \in W_k^{N,p}(\mathbb{R}^2 \times \mathbb{R}^2) \tag{5}$$

*$N \geq 1$ with $p \in (2, \infty)$, $k > 3 - \frac{2}{p}$. Then $\exists$ $(f, u)$ to incompressibleNSVFP on $\mathbb{R}^2 \times \mathbb{R}^2 \times (0, T)$ satisfying*

$$\begin{aligned} & f \in L^\infty(0, T; W_k^{N,p}(\mathbb{R}^2 \times \mathbb{R}^2)), \quad u \in L^\infty(0, T; W^{N,p}(\mathbb{R}^2)), \\ & \langle v \rangle^{\frac{pk}{2}} |\partial_\beta^\alpha f|^{\frac{p-2}{2}} \nabla_v \partial_\beta^\alpha f \in L^2(0, T; L^2(\mathbb{R}^2 \times \mathbb{R}^2)) \quad \text{for } |\alpha| + |\beta| \leq N, \\ & |\partial^\alpha u|^{\frac{p-2}{2}} \nabla_x \partial^\alpha u \in L^2(0, T; L^2(\mathbb{R}^2)) \quad \text{for } |\alpha| \leq N. \end{aligned} \tag{6}$$

# Compressible Navier-Stokes-Vlasov-Fokker-Planck

$$
\begin{aligned}
&\partial_t \rho + \mathrm{div}\,(\rho u) = 0,\\
&\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla p - \Delta u\\
&-\nabla \mathrm{div}\, u = \int_{\mathbb{R}^3} (v-u) F dv,\\
&\partial_t F + v\cdot \nabla_x F + \mathrm{div}_v((u-v)F - \nabla_v F) = 0.
\end{aligned} \tag{7}
$$

Steady state $(\rho_*, 0, \rho_*(2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}})$.

Let $\mu = (2\pi)^{-3/2} e^{-\frac{|v|^2}{2}}$. Set $\rho_* = 1$ and $\sigma = \rho - 1$ $F = \mu + \sqrt{\mu} f$.

# Compressible Navier-Stokes-Vlasov-Fokker-Planck

$$
\begin{aligned}
&\partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\
&\partial_t u - \Delta u - \nabla \mathrm{div}\, u + \frac{p'(\rho)}{\rho}\nabla_x \rho + \frac{1}{\rho}\left( u - \int_{\mathbb{R}^3} v\sqrt{\mu} f dv + u \int_{\mathbb{R}^3} \sqrt{\mu} f dv \right) \\
&\quad = -\frac{\sigma}{\sigma+1}(\Delta u + \nabla \mathrm{div}\, u) - u \cdot \nabla u, \\
&\partial_t f + v \cdot \nabla_x f + u \cdot (\nabla_v f - \frac{v}{2} f) - u \cdot v\sqrt{\mu} = \Delta_v f - \frac{|v|^2}{4} f + \frac{3}{2} f.
\end{aligned} \tag{8}
$$

# Compressible fluid-kinetic equations

**Theorem(M. Chae, K. Kang and L., '12)**

The existence of classical solution to compressible NSVFP equations near equilibrium and exponential convergence when the spatial domain is 3D torus.

# Notation

$$\langle f, g \rangle = \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} f \bar{g} \, dxdv, \quad \langle f, g \rangle_v = \int_{\mathbb{R}^3} f \bar{g} \, dv.$$

$$\partial_\beta^\alpha \equiv \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3}, \qquad \alpha = [\alpha_1, \alpha_2, \alpha_3], \quad \beta = [\beta_1, \beta_2, \beta_3].$$

$$|f|_k^2 = \sum_{|\alpha| \le k} \|\partial^\alpha f\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)}^2, \qquad \|f\|_k^2 = \sum_{|\alpha| + |\beta| \le k} \|\partial^\alpha \partial_\beta f\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)}^2.$$

$$\|(f, u, \sigma)\|_{\mathcal{H}^k}^2 = |f|_k^2 + \|u\|_k^2 + \|\sigma\|_k^2.$$

# Fokker-Planck operator

$$Lf := -\Delta_v f + \frac{|v|^2}{4} f - \frac{3}{2} f. \tag{9}$$

The linear operator $L$ is non-negative
$\langle Lf, f\rangle_v = \int \left|\nabla_v f + \frac{v}{2} f\right|^2 dv.$

# Lemma

**Lemma**

*Let $\Omega$ be either $\mathbb{R}^3$ or $\mathbb{T}^3$. Suppose that $g(x,v) \in L^2(\Omega \times \mathbb{R}^3)$. Then $Lg = 0$ if and only if $g = c\sqrt{\mu}$ for any $c \in L^2_x(\Omega)$. Moreover, there is a $C > 0$ such that*

$$\langle Lg, g\rangle \geq C\|(I - P_0)g\|^2_\sigma, \tag{10}$$

*where inner product and the norm are over $\Omega \times \mathbb{R}^3$.*

$$P_0 : L^2(\mathbb{R}^3_v) \to \mathcal{N} = \langle\sqrt{\mu}\rangle, \qquad P_0 f = \langle f, \sqrt{\mu}\rangle_v \sqrt{\mu}. \tag{11}$$

# A Priori Estimates

$$\frac{1}{2}\frac{d}{dt}\left(\|\partial^\alpha f\|^2 + \|\sqrt{\rho}\partial^\alpha u\|^2 + (\gamma-1)\|\sqrt{\rho}^{\gamma-2}\partial^\alpha \sigma\|^2 + |\overline{\rho u}|^2\right)$$
$$+ \kappa\|\sqrt{\rho}\partial^\alpha \nabla u\|^2 + \kappa'\|\sqrt{\rho}\mathrm{div}\,\partial^\alpha u\|^2 + \|\partial^\alpha u\sqrt{\mu} - \nabla_v \partial^\alpha f - \frac{v}{2}\partial^\alpha f\|^2$$
$$+ (2 - \|\sigma/\rho\|)|\overline{\rho u}|^2 \leq \qquad I_1 + \cdots + I_{10}. \tag{12}$$

# Apriori estimates

$$\langle \partial^\alpha f_t, \partial^\alpha f \rangle + \langle \partial^\alpha (u \cdot (\nabla_v f - \frac{v}{2} f)), \partial^\alpha f \rangle - \underline{\langle \partial^\alpha u \cdot v\sqrt{\mu}, \partial^\alpha f \rangle}$$
$$+ \underline{\|\nabla_v \partial^\alpha f + \frac{v}{2}\partial^\alpha f\|^2} = 0. \tag{13}$$

$$\langle \rho \partial^\alpha u, \partial^\alpha u_t \rangle - \langle \rho \partial^\alpha u, \kappa \Delta \partial^\alpha u + \kappa' \nabla \mathrm{div}\, \partial^\alpha u \rangle + \langle \partial^\alpha u, p'(\rho) \nabla \partial^\alpha \rho \rangle$$
$$+ (\gamma - 1) \langle \rho \partial^\alpha u, \sum_{\beta < \alpha} \partial^{\alpha - \beta} \rho^{\gamma - 2} \partial^\beta \nabla \rho \rangle$$
$$+ \langle \partial^\alpha u, \partial^\alpha (\underline{u - \int v\sqrt{\mu} f dv} + u \int \sqrt{\mu} f dv) \rangle$$
$$+ \cdots . \tag{14}$$

# A priori estimates

$$\|\nabla_v \partial^\alpha f + \frac{v}{2}\partial^\alpha f\|^2 - 2\langle \partial^\alpha u \cdot v\sqrt{\mu}, \partial^\alpha f\rangle + \|\partial^\alpha u\|^2$$
$$= \|\partial^\alpha u\sqrt{\mu} - \nabla_v \partial^\alpha f - \frac{v}{2}\partial^\alpha f\|^2.$$

## Lemma

*If* $\sup_{[0,T]} \|(f,u,\sigma)\|_{\mathcal{H}^3} < \epsilon$, *then*

$$\|(f,u,\sigma)(s)\|^2_{\mathcal{H}^3} + C\int_0^s \|u\|_4^2 + \|\partial_t u\|_2^2 + \|\sigma\|_3^2$$
$$+|(I-P_0)f|_3^2 + |\overline{\rho u}|^2 d\tau \le C\|(f,u,\sigma)(0)\|^2_{\mathcal{H}^3}.$$

To obtain decay estimates, we should compensate the missing dissipation.

**Lemma**

$$\|(f,u,\sigma)(s)\|_{\mathcal{H}^3}^2 + \int_0^s \|u\|_4^2 + \|\partial_t u\|_2^2 + \|\sigma\|_3^2 + |f|_3^2 + |\overline{\rho u}|^2 d\tau \leq C\|(f,u,\sigma)(0)\|_{\mathcal{H}^3}^2 \tag{15}$$

$$\partial_t Im\langle\langle (w)_n, R(\omega)(w)_n\rangle\rangle + |n||(w_1)_n|^2 + Re\langle\langle (w)_n, R(\omega)\bar{L}(w)_n\rangle\rangle \leq Re\langle\langle (w)_n, R(\omega)((\bar{R})_n + (\bar{g})_n)\rangle\rangle + |n|\sum_{i=2}^{4}|(w_i)_n|^2$$